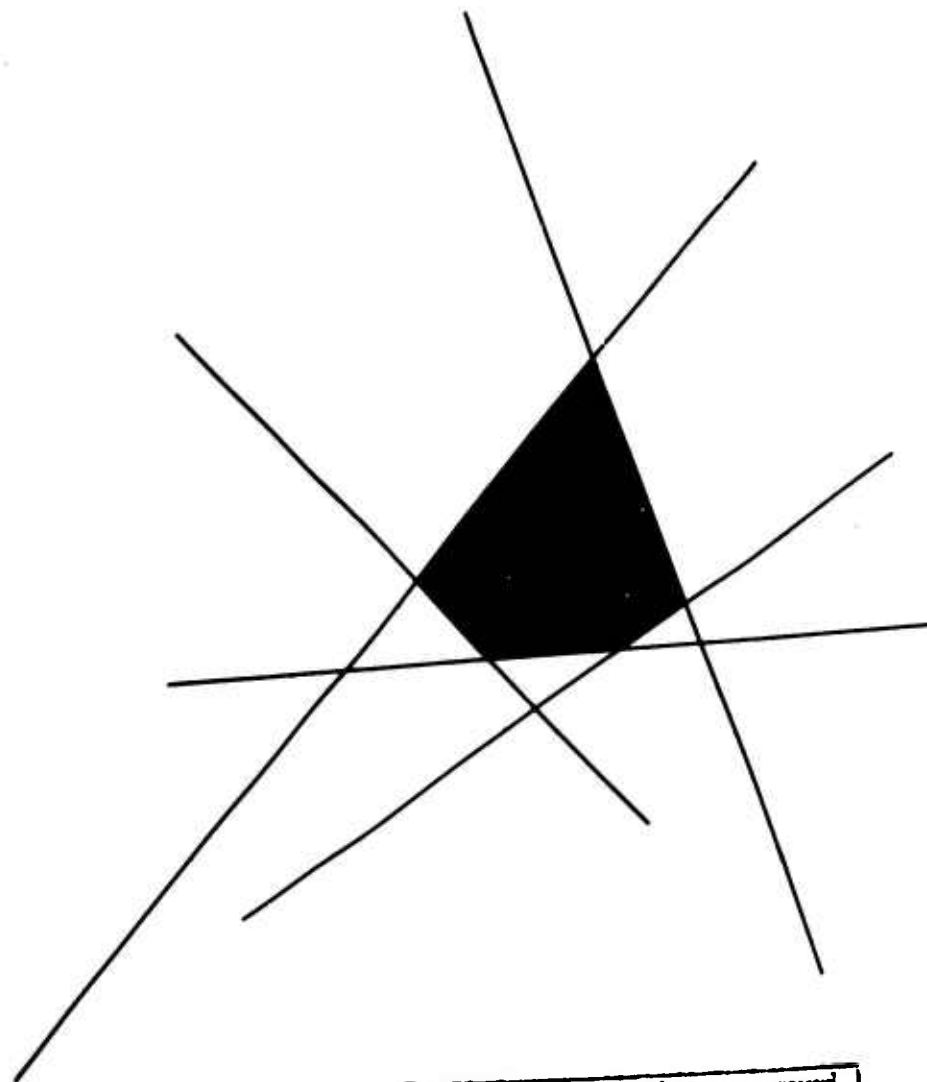


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# INFINITESIMAL LOOK-AHEAD STOPPING RULES

by

SHELDON M. ROSS



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# INFINITESIMAL LOOK-AHEAD STOPPING RULES

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APRIL 1969

ORC 69-7

This research has been partially supported by the Office of Naval Research under Contract Nonr-222(83), the National Science Foundation under Grant GP-8695, and the U. S. Army Research Office-Durham under Contract DA-31-124-ARO-D-331 with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

#### ABSTRACT

The continuous time optimal stopping problem is considered and an infinitesimal look ahead procedure is defined. Sufficient conditions are then given which ensure that this procedure, which is the continuous time analogue of the one stage look ahead rule in the discrete time problem, is optimal. These results are then applied to a class of continuous time Markov decision processes.

# INFINITESIMAL LOOK-AHEAD STOPPING RULES

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## 1. INTRODUCTION

Let  $X = (X_t, t \geq 0)$  be a Markov Process having stationary transition distributions, and sample paths which are almost surely right continuous and have only jump discontinuities. The state space  $S$  of the process is assumed to be a Borel subset of a complete separable metric space and we consider the problem of selecting a stopping time  $\tau$  maximizing

$$(1) \quad E^x \left[ e^{-\lambda \tau} f(X_\tau) - \int_0^\tau e^{-\lambda s} c(X_s) ds \right],$$

where  $f$  and  $c$  are continuous real-valued functions on  $S$ ,  $\lambda \geq 0$ , and  $E^x$  denotes expectation conditional on  $X_0 = x$ .

In the second section of this paper we show that under certain conditions an infinitesimal look-ahead procedure is optimal. This result generalizes certain discrete time results given by Chow-Robbins [4]. In the third section a related approach is described and the resultant procedure is shown to be optimal under slightly more general situations. The fourth section considers a class of continuous time Markovian Decision Processes for which the criterion function is closely related to (1).

## 2. INFINITESIMAL LOOK-AHEAD STOPPING RULE

A stopping time  $\tau$  is defined to be any nonnegative extended real-valued random variable such that for all  $t > 0$ ,  $\{\tau < t\}$  is contained in the sigma field generated by  $\{X_s, 0 \leq s < t\}$ . A stopping time  $\tau^*$  is said to be optimal at  $x \in S$  if  $E^x \left[ e^{-\lambda \tau^*} f(X_{\tau^*}) - \int_0^{\tau^*} e^{-\lambda s} c(X_s) ds \right] = \text{Max}_{\tau} E^x \left[ e^{-\lambda \tau} f(X_{\tau}) - \int_0^{\tau} e^{-\lambda s} c(X_s) ds \right]$ . If  $\tau^*$  is optimal at  $x$  for every  $x \in S$  then it is said to be optimal.

For any Borel set  $B \in S$ , let  $P_t(B | x) = P\{X_t \in B | X_0 = x\}$ , we shall suppose at first that  $\lambda = 0$ . Let

$$(2) \quad \alpha(x) = \lim_{h \rightarrow 0^+} E^x \left[ \frac{f(X_h) - f(x)}{h} \right], \quad x \in S.$$

We assume that  $f$  and  $X$  are such that both the limit in (2) exists, and also that Kolmogorov's forward equation --  $E^x[\alpha(X_t)] = \int f(y) \frac{\partial}{\partial t} P_t(dy | x)$  -- holds (see Breiman [3] page 327).

### Lemma 2.1:

Under certain regularity conditions (given below)

$$(3) \quad E^x \int_0^t \alpha(X_s) ds = E^x f(X_t) - f(x) \quad \forall x \in S, \forall t > 0.$$

### Proof:

We proceed "formally" as follows

$$(4) \quad E^x \int_0^t \alpha(X_s) ds = \int_0^t E^x[\alpha(X_s)] ds$$

$$(5) \quad = \int_0^t \int f(y) \frac{\partial}{\partial s} P_s(dy | x) ds$$

$$(6) \quad = \int f(y) \int_0^t \frac{\partial}{\partial s} P_s(dy | x) dt$$

$$(7) \quad = \int f(y) P_t(dy | x) - f(x)$$

$$(8) \quad = E^x f(X_t) - f(x) .$$

Q.E.D.

Remark:

From the proof it is clear that regularity conditions are necessary for (i) interchanging the integral and expectation in (4), (ii) interchanging the integrals in (6), and (iii) justifying (7). A sufficient condition for (i) is that  $\int_0^t E^x |\alpha(X_s)| ds < \infty \quad \forall x, t$  (see Doob, page 63); Fubini's theorem provides sufficient conditions for (ii); and a continuity condition of the form

$$\lim_{h \rightarrow 0} P_h(\{y\} | x) = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases} \quad \text{is sufficient for (iii).}$$

$$\text{Let } W_t = f(X_t) - \int_0^t \alpha(X_s) ds - f(X_0) .$$

Lemma 2.2:

$$E^x[W_t | X_u, 0 \leq u \leq s] = W_s \quad \text{a.s.} \quad \forall s < t, \forall x .$$

Proof:

$$\begin{aligned} E^x[W_t | X_u, 0 \leq u \leq s] &= W_s + E^x \left[ f(X_t) - f(X_s) - \int_s^t \alpha(X_u) du \mid X_u, 0 \leq u \leq s \right] \\ &= W_s + E^x \left[ f(X_{t-s}) - f(X_0) - \int_0^{t-s} \alpha(X_u) du \mid X_s \right] \\ &= W_s, \text{ by Lemma 3.1 .} \end{aligned}$$

Q.E.D.

$$\text{Let } \tau_{\wedge n} = \min(\tau, n) .$$

Lemma 2.3:

For all stopping times  $\tau$  such that  $E^x W_{\tau \wedge n} \rightarrow E^x W_\tau$

$$(9) \quad E^x \left[ f(X_\tau) - \int_0^\tau c(X_s) ds \right] = E^x \int_0^\tau (\alpha(X_s) - c(X_s)) ds + f(x) .$$

Proof:

From Lemmas 2.1 and 2.2 we have that  $(W_t, t \geq 0)$  is a zero-mean Martingale, and it thus follows from a Martingale systems theorem (see Breiman [2], p. 302) that  $E^x(W_\tau) = 0$  for all bounded  $\tau$ . The result then holds by truncation and passage to the limit.

Q.E.D.

Remark:

When  $E\tau < \infty$  and  $f$  bounded (9) is known as Dynkin's Markov time identity (see [2], p. 376).

We now define the set  $B \in \mathcal{S}$  as follows

$$B \equiv \{x : \alpha(x) - c(x) \leq 0\} .$$

We are now ready for the major theorem. Let  $P^x$  denote probability conditional on  $X_0 = x$ .

Theorem 2.4:

Suppose that  $E^x W_{\tau \wedge n} \rightarrow E^x W_\tau$  for all stopping times  $\tau$  and all  $x$ . If  $B$  is closed in the sense that  $P^x\{\exists t \geq 0 : X_t \notin B\} = 0$  for all  $x \in B$ , and if

$$\tau^* = \inf\{t \geq 0 : X_t \in B\}$$

is finite with Prob. 1, then it is optimal.

Proof:

By Lemma 2.3 we have reduced the problem to one in which there is no reward given for stopping, and there is a cost  $\alpha(x) - c(x)$  per unit time for being

in state  $x$ . The result follows obviously from this.

Q.E.D.

What we have done can perhaps best be described as follows: We define  $\tau^*$ , the infinitesimal look-ahead (ILA) rule, to be the one which stops at state  $x$  iff the infinitesimal look-ahead gain is no greater than the stopping gain. Theorem 2.4 then says that if the set of stopping states is closed (in the sense of Theorem 1) then  $\tau^*$  is optimal. This result is clearly the continuous time analogue of the Chow-Robbins result of optimality of the one-stage look-ahead rule in the monotonic case (see [4]).

Example 1:

Let  $Y_1, Y_2, \dots$  be a sequence of iid random variables with cdf  $F^\dagger$ , and let  $(N_t, t \geq 0)$  be a nonhomogeneous Poisson Process, independent of the  $Y_i$ 's, and with a continuous nonincreasing rate function  $\mu(t)$ . Let  $M_t = \max(Y_1, \dots, Y_{N_t})$ , and consider the Markov Process  $\{X_t = (t, M_t), t \geq 0\}$ . We take  $f(t, m) = m$ , and assume that  $c(t, m)$  is nondecreasing in both  $t$  and  $m$ . This is, of course, the continuous time analogue of the famous house-selling problem (though for the sake of generality we have not required that  $F(0) = 0$ , see [4] and [9]).

$$\begin{aligned} \alpha(t, m) &= \lim_{h \rightarrow 0} E \left[ \frac{M_{t+h} - M_t}{h} \mid M_t = m \right] \\ &= \mu(t) E[\max(Y_i, m) - m] \\ &= \mu(t) \int y dF(y + m). \end{aligned}$$

Since  $\int_y y dF(y + m)$  is nonincreasing in  $m$  it follows from Theorem 2.4 that

$$\tau^* = \inf\{t \geq 0 : \mu(t) \int y dF(y + M_t) \leq c(t, M_t)\}$$

is optimal.

---

$^\dagger \int |x| dF(x) < \infty$ .

Example 2:

Let  $(N_t, t \geq 0)$  be a Poisson Process with rate  $\mu$ , and consider the Markov Process  $(X_t = (t, N_t), t \geq 0)$ . Let  $c(t, N_t) = N_t$  and  $g(t, N_t) = -\lambda(T - t)^2/2$ , where  $T$  is some fixed constant. Then  $\alpha(t, N_t) = \lambda(T - t)$  and from Theorem 2.4 it follows that  $\tau^* = \inf\{t \geq 0 : N_t \geq \lambda(T - t)\}$  is optimal. This problem arises in determining the optimal intermediate time to dispatch a Poisson Process (see Ross [10]).

Generalizing the previous results to include the case of a discount factor  $\lambda > 0$  is easy. We consider a new Markov Process  $X'_t$  consisting of the pair  $X'_t = (t, X_t)$ . Letting

$$(10) \quad \begin{aligned} f'(t, x) &= e^{-\lambda t} f(x) \\ c'(t, x) &= e^{-\lambda t} c(x) \end{aligned}$$

we have that

$$(11) \quad \begin{aligned} \alpha'(t, x) &\equiv \lim_{h \rightarrow 0} E^{t, x} \left[ \frac{f(X'_h) - f(t, x)}{h} \right] \\ &= \lim_{h \rightarrow 0} E^x \left[ \frac{e^{-\lambda(t+h)} f(X_h) - e^{-\lambda t} f(x)}{h} \right] \\ &= e^{-\lambda t} \lim_{h \rightarrow 0} E^x \left[ \frac{e^{-\lambda h} f(X_h) - f(x)}{h} \right]. \end{aligned}$$

Thus, letting

$$\begin{aligned} B_\lambda &= \{(t, x) : \alpha'(t, x) - c(t, x) \leq 0\} \\ &= \{x : \alpha_\lambda(x) - c(x) \leq 0\}, \end{aligned}$$

where

$$(12) \quad \alpha_\lambda(x) = \lim_{h \rightarrow 0} E^x \left[ \frac{e^{-\lambda h} f(X_h) - f(x)}{h} \right]$$

we have from Theorem 2.4 that

Corollary 2.5:

If  $B_\lambda$  is closed, and if

$$(13) \quad \tau^* = \inf\{t \geq 0 : X_t \in B_\lambda\}$$

is finite with probability 1, then  $\tau^*$  maximizes (1) for all  $x$ .<sup>†</sup>

Example 3:

Let  $(N_t, t \geq 0)$  be a nonhomogeneous Poisson Process with rate  $\mu(t)$ . Suppose the reward for stopping when  $N_t = x$  is  $x$ , and the continuation rate at  $N_t = x$  is  $c(x)$ . Suppose further that  $\mu(t)$  is continuous nonincreasing, and  $c(x)$  is continuous nondecreasing, and let  $\lambda$  be the discount factor.

The state space is thus  $X_t = (t, N_t)$ , and

$$\begin{aligned} \alpha_\lambda(t, n) &= \lim_{h \rightarrow 0} E^{t, n} \left[ \frac{e^{-\lambda h} N_h - N_0}{h} \right] \\ &= \lim_{h \rightarrow 0} \frac{e^{-\lambda h} (n + \mu(t)h) - n}{h} \\ &= -n\lambda + \mu(t). \end{aligned}$$

Thus from Corollary 2.5 we have that

$$\tau^* = \inf \left\{ t \geq 0 : N_t \geq \frac{\mu(t) - c(N_t)}{\lambda} \right\}$$

is optimal. This example with  $c(x) \equiv c$ , and  $\mu(t) \equiv \mu$  was treated in Taylor [12] by a different method.<sup>††</sup>

<sup>†</sup>The regularity condition  $E^{x, \lambda} W_{\tau, n}^\lambda \rightarrow E^{x, \lambda} W_\tau^\lambda \quad \forall x, \quad \forall \tau$  must of course be satisfied, where  $W_t^\lambda$  is appropriately defined.

<sup>††</sup>Taylor's answer differs somewhat from ours as he supposed that

$$[\log_e(1 + \lambda/\mu)]^{-1} - c/\lambda \text{ was an integer.}$$

Example 4:

Consider Example 1 when (i) a discount factor  $\lambda$  is present, (ii)  $\mu(t) \equiv \mu$ , and (iii)  $c(t,m) \equiv c$ . Then it is easily seen that

$$\tau^* = \inf\{t \geq 0 : \mu \int y dF(y + m) - \lambda M_t \leq c\}$$

is optimal.

Now, consider the same problem with the exception that once an offer is rejected it is no longer available. Clearly, the optimal return for this problem is no greater than the optimal return for the original problem. Thus, since the optimal policy  $\tau^*$  is a legitimate policy for this new problem (as it never accepts an old offer) it follows that it is also optimal for this problem. This is related to certain results given by Elfving [7], and Siegmund [13].

Comment:

We note here that any general discount function  $d(t)$  may be handled in the same manner as we dealt with  $e^{-\lambda t}$ .

### 3. A RELATED APPROACH

Let  $Z_t = e^{-\lambda t} f(X_t) - \int_0^t e^{-\lambda s} c(X_s) ds$ , and let  $\bar{B} = \{x \in S : E^x Z_t \leq f(x) \text{ for all } t \geq 0\}$ . Thus,  $\bar{B}$  is the set of states at which stopping is better than continuing for any *fixed* amount of time.

#### Lemma 3.1:

If  $P^x\{\exists t \geq 0 : X_t \notin \bar{B}\} = 0$  for all  $x \in \bar{B}$ , then

$$E^x[Z_t \mid X_u, 0 \leq u \leq s] \leq Z_s \text{ a.s. } \forall x \in \bar{B}, \forall s < t.$$

#### Proof:

$$\begin{aligned} E^x[Z_t \mid X_u, 0 \leq u \leq s] &= Z_s + e^{-\lambda s} E^x \left[ e^{-\lambda(t-s)} f(X_t) - \int_s^t e^{-\lambda(u-s)} c(X_u) du - f(X_s) \mid X_s \right] \\ &= Z_s + e^{-\lambda s} E^x \left[ e^{-\lambda(t-s)} f(X_{t-s}) - \int_0^{t-s} e^{-\lambda u} c(X_u) du - f(X_0) \mid X_s \right]. \end{aligned}$$

Since  $x \in \bar{B}$  implies by hypothesis that  $X_s \in \bar{B}$  a.s., the result follows from the definition of  $\bar{B}$ .

Q.E.D.

#### Lemma 3.2:

If  $\lim_{n \rightarrow \infty} E^x Z_{\tau, n} = E^x Z_\tau \quad \forall x, \forall \tau$ , and if

$$P^x\{\exists t \geq 0 : X_t \notin \bar{B}\} = 0 \quad \forall x \in \bar{B},$$

then

$$E^x Z_\tau \leq f(x) \quad \forall x \in \bar{B}, \forall \tau.$$

Proof:

It follows from Lemma 3.1 that  $(Z_t, t \geq 0)$  is a supermartingale, and so the result obtains from a standard supermartingale systems theorem as in Lemma 2.3.

Q.E.D.

We have thus shown the following:

Theorem 3.3:

Under the conditions of Lemma 3.2

$$\sup_{\tau} E^x(Z_{\tau}) = \begin{cases} f(x) & x \in \bar{B} \\ > f(x) & x \notin \bar{B} . \end{cases}$$

Proof:

When  $x \in \bar{B}$ ,  $\sup = f(x)$  from Lemma 3.2. When  $x \notin \bar{B}$ , the result follows from the definition of  $\bar{B}$ .

Q.E.D.

Let us define the stopping time  $\bar{\tau}$  by

$$\bar{\tau} = \inf \{t \geq 0 : X_t \in \bar{B}\} .$$

Now, suppose that there exists an optimal stopping rule  $\tau$  for  $x$ . Let  $\tau_1 = \min(\tau, \bar{\tau})$ , then from Lemma 3.2 it follows that  $\tau_1$  is also optimal for  $x$ .<sup>†</sup> But it is easily seen that  $\tau_1$  must be a.s. equal to  $\bar{\tau}$ . Thus if there exists an optimal rule for each  $x$ , then it follows that  $\bar{\tau}$  is optimal. It should also be noted that  $\bar{\tau}$  is just the continuous time analogue of the functional equation rule (see Bellman [1]).

---

<sup>†</sup>All of this is assuming, of course, the conditions of Lemma 3.2.

Some sufficient conditions for the existence of an optimal stopping rule are given in Dynkin [6] and Taylor [12]. To determine the connection between  $\bar{\tau}$  and TLA  $\tau^*$  (as given by [13]), we first note that  $\bar{B} < B_\lambda$  and so  $\bar{\tau} \geq \tau^*$ . To go the other way, we need the following:

Corollary 3.4:

If the conditions of Corollary 2.5 hold, then

$$\bar{\tau} = \tau^*.$$

Proof:

We give the proof for  $\lambda = 0$ . If  $x \in B$ , then since  $B$  is closed it follows that  $E^x \int_0^t (\alpha(X_s) - c(X_s)) ds \leq 0$  for all  $t$ , and so the result follows from (9). Similar comments hold when  $\lambda > 0$ .

Q.E.D.

Aside from its own interest, the reason we have considered this approach is that one may easily construct examples in which  $\bar{B}$  is closed but  $B$  is not. The idea is also illuminating, and we paraphrase it as follows: Call a state bad if stopping at that state is better than continuing from that state for any *fixed* amount of time. Then if this set of states is closed and if an optimal rule exists, then the rule which stops the first time it enters a bad state is optimal. Since a discrete time Markov Process may be regarded as a continuous time Markov Process (with  $X'_t = (t, X_t)$ ), it follows that this result also holds for the discrete time problem.

#### 4. SOME RELATED CRITERIA

In this section we consider the problem of choosing a stopping time  $\tau$  maximizing either

$$(14) \quad \psi_\tau = \frac{E^x \left[ e^{-\lambda\tau} f(X_\tau) - \int_0^\tau e^{-\lambda s} c(X_s) ds \right]}{E^x [1 - e^{-\lambda\tau}]}, \text{ where } \lambda > 0$$

or

$$(15) \quad \phi_\tau = \frac{E^x \left[ f(X_\tau) - \int_0^\tau c(X_s) ds \right]}{E^x \tau}, \text{ where } 0 < E^x \tau < \infty.$$

Criterion (14) represents expected total discounted return, and (15) the long run average cost per unit time, when a sequence of independent stopping games are played, each starting at  $x$ . These criteria also arise in connection with a 2-action, continuous time Markovian Decision Process (see [11]) in which the "stop" action resets the process to a fixed initial state  $x$ . (In this connotation  $-f(y)$  is usually thought of as the cost of resetting from state  $y$ ). For any constant  $b$ , let

$$(16) \quad \psi_\tau(b) = (\psi_\tau - b)E^x[1 - e^{-\lambda\tau}]$$

$$(17) \quad = E^x \left[ e^{-\lambda\tau} f(X_\tau) - \int_0^\tau e^{-\lambda s} (c(X_s) + \lambda b) ds \right],$$

and let

$$(18) \quad \phi_\tau(b) = (\phi_\tau - b)E^x \tau$$

$$(19) \quad = E^x \left[ f(X_\tau) - \int_0^\tau (c(X_s) + b) ds \right].$$

Lemma 4.1:

- (i) If for some  $b$ ,  $0 = \psi_{\star}(b) = \max_{\tau} \psi_{\tau}(b)$ , then  $b = \psi_{\star} = \max_{\tau} \psi_{\tau}$ , and conversely;
- (ii) If for some  $b$ ,  $0 = \phi_{\star}(b) = \max_{\tau \in \Gamma} \phi_{\tau}(b)$  where  $\Gamma = \{\tau : 0 < E^x \tau < \infty\}$ , then  $b = \phi_{\star} = \max_{\tau \in \Gamma} \phi_{\tau}$ , and conversely.

Proof:

Follows directly from (16) and (18).

Remark:

Part (i) of the above Lemma seems to be new as criterion (14) does not seem to have been previously considered in stopping rule literature. Part (ii) is not new and may be found in either Breiman [2] or Taylor [12].

We shall suppose for the remainder that optimal rules exist for criterions (14) and (15) and we let  $V = \max_{\tau} \psi_{\tau}$ , and  $g = \max_{\tau \in \Gamma} \phi_{\tau}$ .

Theorem 4.2:

Under the usual regularity conditions

- (i) If  $B_1 = \{x : \alpha_{\lambda}(x) - c(x) \leq V\lambda\}$  is closed, then  $\tau^{\star} = \inf\{t \geq 0 : X_t \in B_1\}$  is optimal for (14).
- (ii) If  $B_2 = \{x : \alpha(x) - c(x) \leq g\}$  is closed, then  $\tau^{\star} = \inf\{t \geq 0 : X_t \in B_2\}$  is optimal for (15) whenever  $0 < E^x \tau^{\star} < \infty$ .

Proof:

Follows directly from (17), (19), Corollary 2.5 and Lemma 4.1.

Q.E.D.

Remark:

Since  $V$  and  $g$  are in general unknown the usefulness of Theorem 4.2 is

mainly that it enables us to determine the structure of the optimal rule.

Example 5:

Let  $(N_t, t \geq 0)$  be any right continuous counting process with left limits. Let  $c(n)$  be the cost rate when there are  $n$  in the system, and suppose that  $c(n)$  is nondecreasing. Let  $f(x) \equiv -R$  (i.e.,  $R$  is the reset cost). Then the related Markov Process is

$$X_t = (N_s, s \leq t),$$

and

$$\alpha_\lambda(X_t) = \lambda R, \quad \alpha(X_t) = 0$$

$\tau^* = \inf\{t \geq 0 : c(N_t) \geq \lambda(R - V)\}$  is optimal for (14), and

$\tau^* = \inf\{t \geq 0 : c(N_t) \geq -g\}$  is optimal for (15). Thus, for the average cost case it is optimal to reset the process whenever the present cost rate is at least as large as the optimal average cost per unit time.

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Unclassified

Security Classification

| DOCUMENT CONTROL DATA - R & D  |   |  |
|--|---|--|
| (Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)   |   |  |
| 1. ORIGINATING ACTIVITY (Corporate author)   |   | 2a. REPORT SECURITY CLASSIFICATION   |
| University of California, Berkeley   |   | Unclassified   |
|  |   | 2b. GROUP  |
|  |   |  |
| 3. REPORT TITLE  |   |  |
| INFINITESIMAL LOOK-AHEAD STOPPING RULES  |   |  |
| 4. DESCRIPTIVE NOTES (Type of report and inclusive dates)  |   |  |
| Research Report  |   |  |
| 5. AUTHOR(S) (First name, middle initial, last name)   |   |  |
| Sheldon M. Ross  |   |  |
| 6. REPORT DATE   | 7a. TOTAL NO. OF PAGES  | 7b. NO. OF REFS  |
| April 1969   | 15  | 13   |
| 8a. CONTRACT OR GRANT NO.  | 9a. ORIGINATOR'S REPORT NUMBER(S)   |  |
| DA-31-124-ARO-D-331  | ORC 69-7  |  |
| b. PROJECT NO.   |   |  |
| 20014501B14C   |   |  |
| c.   | 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) |  |
| d.   |   |  |
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| Also supported by the Office of Naval Research under Contract Nonr-222(83) and the Nat'l. Sci. Found. under Grant GP-8695.   |   | U.S. Army Research Office-Durham<br>Box CM, Duke Station<br>Durham, North Carolina 27706 |
| 13. ABSTRACT   |   |  |
| <p>↓</p> <p>The continuous time optimal stopping problem is considered and an infinitesimal look ahead procedure is defined. Sufficient conditions are then given which ensure that this procedure, which is the continuous time analogue of the one stage look ahead rule in the discrete time problem, is optimal. These results are then applied to a class of continuous time Markov decision processes.</p> |   |  |

Unclassified

Security Classification

| 14<br>KEY WORDS   | LINK A |    | LINK B |    | LINK C |    |
|---|--------|----|--------|----|--------|----|
|   | ROLE   | WT | ROLE   | WT | ROLE   | WT |
| Optimal Stopping<br>Infinitesimal Look Ahead<br>Martingale<br>Markov Decision Process |        |    |        |    |        |    |